9 BINARY RELATIONS

We find pairs of objects all around us: wives and their husbands, friends and their telephone numbers, students and the subjects they study, customers and their bank accounts, guests and their hotel reservations. We see that a binary relation is just a set of ordered pairs. We see how the first and second functions split an ordered pair into its coordinates. We look at the domain and range, and the source and target, of a binary relation. We see how domain and range restriction work like database queries. We see how relational image works like a table lookup. We see how the override operator works like a database update. We look at the inverse function. We see how two binary relations may be composed to form a third binary relation.

9.1 BINARY RELATIONS

A binary relation is just a set of pairs.

$$\{ \overbrace{(madge, homer)}^{\text{a pair}}, (wilma, fred) \}$$

$$\{ (kylie, 1), (kylie, 2), (robbie, 3) \}$$

$$\{ (sue, formalMethods), (sam, formalMethods), (tim, maths) \}$$

We can use the usual set operations - union, intersection, difference and cardinality - on sets of binary pairs.

$$\{ (1, a), (2, b) \} \cup \{ (2, b), (3, c) \} = \{ (1, a), (2, b), (3, c) \}$$

$$\{ (1, a), (2, b) \} \cap \{ (2, b), (3, c) \} = \{ (2, b) \}$$

$$\{ (1, a), (2, b) \} \setminus \{ (2, b) (3, c) \} = \{ (1, a) \}$$

$$\#\{ (1, a), (3, b), (5, c) \} = 3$$

Each element in a binary relation is a pair of objects.

EXERCISE 9.1

1 Describe three further examples of binary relations.

2 Evaluate

a $\{ (a, x), (b, y) \} \cup \{ (a, z), (b, w) \}$
b $\{ (a, x), (b, y) \} \cup \{ (b, y), (a, x), (c, z) \}$
c $\{ (a, x), (b, y) \} \cap \{ (a, z), (b, y) \}$
d $\{ (a, x), (b, y) \} \setminus \{ (a, x), (b, w) \}$
e $\#\{ (a, x), (b, y), (c, z), (d, w) \}$

9.2 THE PROJECTION FUNCTIONS

Each element in a binary relation is a pair of objects, e.g. (*madge, homer*).

(*madge, homer*) is not the same pair as (*homer, madge*); order matters.

The ordered pair (*madge, homer*) may be written as $madge \mapsto homer$. This is known as *maplet notation*, and $madge \mapsto homer$ is known as a *maplet*.

$$(madge, homer) = madge \mapsto homer$$

In general, we use maplet notation to represent an element in a binary relation; we use coordinate notation to represent a pair by itself.

$$partners = \{ madge \mapsto homer, wilma \mapsto fred \}$$
$$aPair = (madge, homer)$$

The Z functions *first* and *second* split an ordered pair into its first and second coordinates.

$$first\ (madge, homer) = madge$$
$$second\ (madge, homer) = homer$$

first and *second* are known as the *projection functions* for ordered pairs.

EXERCISE 9.2

1 Evaluate $(first\ (a, 1), second\ (a, 1))$

9.3 DOMAIN AND RANGE

Look at this binary relation:

$\{ 1 \mapsto a, 2 \mapsto b, 3 \mapsto c \}$

The set formed by all the first coordinates

$\{ 1, 2, 3 \}$

is known as its *domain*. We write

$\text{dom}\ \{ 1 \mapsto a, 2 \mapsto b, 3 \mapsto c \} = \{ 1, 2, 3 \}$

where dom is Z for domain.

The set formed by all the second coordinates

$\{ a, b, c \}$

is known as its range. We write

$\text{ran}\ \{ 1 \mapsto a, 2 \mapsto b, 3 \mapsto c \} = \{ a, b, c \}$

where ran is Z for range.

EXERCISE 9.3

1 Evaluate

a dom { *A*301 ↦ *office*, *A*302 ↦ *classRoom*, *A*303 ↦ *lab*, *A*304 ↦ *classRoom* }

b ran { *A*301 ↦ *office*, *A*302 ↦ *classRoom*, *A*303 ↦ *lab*, *A*304 ↦ *classRoom* }

c dom { *terry* ↦ *Access*, *garry* ↦ *ECDL*, *kanti* ↦ *ALevel*, *kanti* ↦ *HND* }

d ran { *terry* ↦ *Access*, *garry* ↦ *ECDL*, *kanti* ↦ *ALevel*, *kanti* ↦ *HND* }

e dom { *richard* ↦ 1921, *richardII* ↦ 1952, *richardIII* ↦ 1976 }

f ran { *richard* ↦ 1921, *richardII* ↦ 1952, *richardIII* ↦ 1976 }

9.4 SOURCE AND TARGET

Look at the binary relation $\{ 1 \mapsto a, 2 \mapsto b, 3 \mapsto c \}$. We can see that

$$\text{dom} \{ 1 \mapsto a, 2 \mapsto b, 3 \mapsto c \} = \{ 1, 2, 3 \}$$

and

$$\{ 1, 2, 3 \} \subseteq \mathbb{Z}$$

$\mathbb{Z}$ is known as the *source*. The domain of a binary relation is a subset of its source.

Now we define CHARACTER - the set of all characters of the alphabet:

$$[\ CHARACTER\ ]$$

We can see that

$$\text{ran} \{ 1 \mapsto a, 2 \mapsto b, 3 \mapsto c \} = \{ a, b, c \}$$

and

$$\{ a, b, c \} \subseteq CHARACTER$$

CHARACTER is known as the *target*. The range of a binary relation is a subset of its target.

9.5 DECLARING A BINARY RELATION

We introduce as given types the set of all dates and the set of all people.

[*DATE, PERSON*]

We define *appointments* as a binary relation from *DATE* to *PERSON* like this:

$appointments : DATE \leftrightarrow PERSON$

An example set of appointments is:

$appointments = \{ 7Nov \mapsto tom, 7Nov \mapsto ann, 8Nov \mapsto jerry \}$

$\leftrightarrow$ is the binary relation operator. Even though it looks like a double-headed arrow, it associates from left to right.

Naming a binary relation is easy if you can describe the relationship between the first and second element of each pair. For example:

- lecturers teach subjects $teaches : LECTURER \leftrightarrow SUBJECT$ e.g. marris teaches Z
- owners have motor vehicles $owns : OWNER \leftrightarrow VEHICLE$ e.g. marris owns BMW320
- parents give birth to children: $hasChild : PERSON \leftrightarrow PERSON$ e.g. albert hasChild harold

EXERCISE 9.4

1 Introduce appropriate types and declare the binary relations described below, and give an example of an element in each binary relation.

a debtors and the money they owe

b driving instructors and their pupils

c owners and their horses

d students and the subjects they study

e authors and the titles of the books they have had published

9.6 DOMAIN RESTRICTION

Domain restriction works like a database query, a bit like using an internet search engine to list all entries that contain the phrase "Z Notation". You might get many entries listed, or none at all.

Look at this binary relation:

$$\{ 1 \mapsto a, 2 \mapsto b, 3 \mapsto c \}$$

Its domain is { 1, 2, 3 }

If we restrict the binary relation so that its domain is { 2 } we get

$$\{ 2 \mapsto b \}$$

We write

$$\{ 2 \} \lhd \{ 1 \mapsto a, 2 \mapsto b, 3 \mapsto c \} = \{ 2 \mapsto b \}$$

where $\lhd$ means *domain restriction*. $\lhd$ is known as the *domain restriction operator*.

The domain restriction operator defines a subset of a given binary relation. For example:

$$\{ 4 \} \lhd \{ 1 \mapsto a, 2 \mapsto b, 3 \mapsto c \} = \{ \}$$

$$\{ 1, 3 \} \lhd \{ 1 \mapsto a, 2 \mapsto b, 3 \mapsto c \} = \{ 1 \mapsto a, 3 \mapsto c \}$$

9.7 RANGE RESTRICTION

Look again at this binary relation:

$\{ 1 \mapsto a, 2 \mapsto b, 3 \mapsto c \}$

Its range is $\{ a, b, c \}$.

If we restrict the binary relation so that its range is $\{ b \}$ we get

$\{ 2 \mapsto b \}$

We write

$\{ 1 \mapsto a, 2 \mapsto b, 3 \mapsto c \} \rhd \{ b \} = \{ 2 \mapsto b \}$

where $\rhd$ means *range restriction*. $\rhd$ is known as the range *restriction operator*.

The range restriction operator defines a subset of a given binary relation. For example:

$\{ 1 \mapsto a, 2 \mapsto b, 3 \mapsto c \} \rhd \{ d \} = \{ \}$

$\{ 1 \mapsto a, 2 \mapsto b, 3 \mapsto c \} \rhd \{ a, c \} = \{ 1 \mapsto a, 3 \mapsto c \}$

EXERCISE 9.5

1 Evaluate

a $\{ A, C \} \lhd \{ A \mapsto carrots, B1 \mapsto pasta, C \mapsto potatoes, C \mapsto tomatoes, D \mapsto milk \}$

b $\{ red \mapsto 1, brown \mapsto 2, green \mapsto 3, yellow \mapsto 4, blue \mapsto 5, pink \mapsto 6, black \mapsto 7 \} \rhd 3..5$

c $\{ king \} \lhd \{ pawn \mapsto 1, rook \mapsto 5, knight \mapsto 3, bishop \mapsto 3, queen \mapsto 9 \}$

d $\{ pawn \mapsto 1, rook \mapsto 5, knight \mapsto 3, bishop \mapsto 3, queen \mapsto 9 \} \rhd \{ n : \mathbb{Z} \mid n > 3 \}$

9.8 RELATIONAL IMAGE

The relational image operator works like a table look-up. For example, look at this table:

colour	***value***
red	1
brown	4
green	3
yellow	2
blue	5
pink	6
black	7

You ask: what is blue's numerical value? You look down the colour column for blue, and then across see that blue's numerical value is 5.

Look at this binary relation:

$$\{ 1 \mapsto a, 2 \mapsto b, 3 \mapsto c \}$$

What is the second coordinate of the pair whose first coordinate is 2? The answer is *b*.

We write:

$$\{ 1 \mapsto a, 2 \mapsto b, 3 \mapsto c \} \llparenthesis \{ 2 \} \rrparenthesis = \{ b \}$$

where the fat brackets $\llparenthesis \ldots \rrparenthesis$ is the relational image operator. You provide a subset of a binary relation's source, it gives you the corresponding elements in the binary relation's range. For example:

$$\{ 1 \mapsto a, 2 \mapsto b, 3 \mapsto c \} \llparenthesis \{ 4 \} \rrparenthesis = \{ \}$$

Input from domain

$$\{ 1 \mapsto a, 2 \mapsto b, 3 \mapsto c \} \llparenthesis \{ 1, 3 \} \rrparenthesis = \{ a, c \}$$

Output from range

EXERCISE 9.6

1 Evaluate

a $\{ \mathit{eatsMeat} \mapsto \mathit{tom}, \mathit{vegetarian} \mapsto \mathit{ann}, \mathit{eatsMeat} \mapsto \mathit{jerry} \} \llparenthesis \{ \mathit{eatsMeat} \} \rrparenthesis$

b $\{ 65 \mapsto A, 66 \mapsto B, 67 \mapsto C, 68 \mapsto D, 69 \mapsto E \} \llparenthesis \{ 65, 67, 69 \} \rrparenthesis$

c $\{ \mathit{tea} \mapsto 50, \mathit{coffee} \mapsto 75, \mathit{hotChocolate} \mapsto 75 \} \llparenthesis \{ \mathit{soup} \} \rrparenthesis$

d $\{ \mathit{tom} \mapsto 12{:}00, \mathit{ann} \mapsto 12{:}20, \mathit{jerry} \mapsto 12{:}40 \} \llparenthesis \{ \mathit{ann} \} \rrparenthesis$

e $\{ 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 5, 4 \mapsto 7, 5 \mapsto 11 \} \llparenthesis 1..5 \rrparenthesis$

2 If $R = \{ 1 \mapsto a, 2 \mapsto b, 3 \mapsto c, 4 \mapsto d, 5 \mapsto e \}$ and $S = 2..4$ evaluate:

a $R \llparenthesis S \rrparenthesis$

b $\mathrm{ran}(S \lhd R)$

What do you notice about your answers to 2a and 2b above?

9.9 OVERRIDE

The override operator, $\oplus$, works like a database update. For example:

$\{ 1 \mapsto a, 2 \mapsto b, 3 \mapsto c \} \oplus \{ 2 \mapsto B \} = \{ 1 \mapsto a, 2 \mapsto B, 3 \mapsto c \}$	$2 \mapsto b$ is replaced with $2 \mapsto B$
$\{ 1 \mapsto a, 2 \mapsto b, 3 \mapsto c \} \oplus \{ 4 \mapsto d \} = \{ 1 \mapsto a, 2 \mapsto b, 3 \mapsto c, 4 \mapsto d \}$	$4 \mapsto d$ has been added

EXERCISE 9.7

1 Evaluate

a $\{ \mathit{tea} \mapsto 50, \mathit{coffee} \mapsto 75, \mathit{hotChocolate} \mapsto 75 \} \oplus \{ \mathit{tea} \mapsto 55 \}$

b $\{ \mathit{smith} \mapsto \mathit{beer}, \mathit{kline} \mapsto \mathit{cider}, \mathit{french} \mapsto \mathit{wine} \} \oplus \{ \mathit{jack} \mapsto \mathit{whisky} \}$

c $\{ \mathit{andy} \mapsto \mathit{mon}, \mathit{bettie} \mapsto \mathit{tue}, \mathit{carol} \mapsto \mathit{wed} \} \oplus \{ \mathit{carol} \mapsto \mathit{thu}, \mathit{fred} \mapsto \mathit{wed} \}$

9.10 INVERSE

We introduce as a given type DRINK, the set of all drinks.

$[\ DRINK\ ]$

We define costs as a binary relation from drink to price.

$$costs : DRINK \leftrightarrow \mathbb{Z}$$

$$costs = \{\ tea \mapsto 50,\ coffee \mapsto 75,\ hotChocolate \mapsto 75,\ soup \mapsto 75\ \}$$

So, for example, tea costs 50 pence.

If we reversed the binary relation, if we made it from price to drink and called it buys, so, for example, 50 pence buys tea:

$$buys : \mathbb{Z} \leftrightarrow DRINK$$

then

$$buys = \{\ 50 \mapsto tea,\ 75 \mapsto coffee,\ 75 \mapsto hotChocolate,\ 75 \mapsto soup\ \}$$

buys is the *inverse* of *costs*. We write:

$$costs\sim\ = buys$$

(say costs inverse equals buys). ~ (tilde) is the inverse operator. The inverse of a binary relation is another binary relation with the coordinates of their ordered pairs reversed.

EXERCISE 9.8

1 We introduce the type *NAME*, the set of all people's names, and define

$$\begin{array}{|l} alias : NAME \leftrightarrow NAME \\ \hline alias = \{ tom \mapsto dopey, sam \mapsto dozy, stu \mapsto dreamy, tom \mapsto sleepy, sam \mapsto titch \} \end{array}$$

Evaluate:

a $alias^{\sim}$

b $alias \rhd \{ dozy, dreamy \}$

c $alias ⦇ \{ tom, sam \} ⦈$

d $alias ⦇ \{ sue \} ⦈$

e $alias^{\sim} ⦇ \{ dreamy, dozy \} ⦈$

9.11 COMPOSITION

Given the types

$[\ PERSON, ROOM\ ]$

the set of all persons and the set of all rooms in a department respectively, look at the two binary relations, *hasPhone* and *phoneInRoom*, shown below.

$$hasPhone : PERSON \leftrightarrow \mathbb{Z}$$

$$hasPhone = \{\ roy \mapsto 317, tom \mapsto 208, tom \mapsto 209, jim \mapsto 326, lee \mapsto 225\ \}$$

$$phoneInRoom : \mathbb{Z} \leftrightarrow ROOM$$

$$phoneInRoom = \{\ 317 \mapsto A306, 208 \mapsto A309, 209 \mapsto A309, 326 \mapsto A306, 225 \mapsto A309\ \}$$

roy has phone 317; phone 317 is in room *A*306. Therefore, we can conclude that *roy* is in room *A*306. Starting from *PERSON* we can reach *ROOM* via $\mathbb{Z}$ (their phone number).

The composition of two binary relations is another binary relation where the range of one is a subset of the domain of the other.

$$\mathrm{ran}\ hasPhone \subseteq \mathrm{dom}\ phoneInRoom$$

We write:

$$hasPhone \fatsemi phoneInRoom = \{\ roy \mapsto A306, tom \mapsto A309, jim \mapsto A306, lee \mapsto A309\ \}$$

where $\fatsemi$ is called the *sequential composition operator*.

EXERCISE 9.9

1 In a horse trials competition, an owner may enter more than one horse and a rider may ride more than one horse. Given the types [*HORSE*, *OWNER*, *RIDER*] and the binary relations

$$entered : OWNER \leftrightarrow HORSE$$

$$entered = \{\ sam \mapsto merryTom,\ sam \mapsto jumpingJack,\ pam \mapsto hissingSid,\ jan \mapsto ticTac,\ tel \mapsto isAGas\ \}$$

$$riddenBy : HORSE \leftrightarrow RIDER$$

$$riddenBy = \{\ merryTom \mapsto jones,\ ticTac \mapsto jan,\ jumpingJack \mapsto french,\ isAGas \mapsto jan,\ hissingSid \mapsto fraser\ \}$$

write down the contents of *ridesFor* when $ridesFor = riddenBy^{\sim} \mathbin{⨾} entered^{\sim}$.

REVIEW

We find pairs of objects all around us: wives and their husbands, friends and their telephone numbers, students and the subjects they study, customers and their bank accounts, guests and their hotel reservations. We noted that a binary relation is just a set of ordered pairs. We saw how the first and second functions extracted the first and second coordinates of an ordered pair. We saw how domain and range restriction work like database queries. We saw how relational image works like a table lookup. We saw how the override operator works like database update. We noted that inverse reverses the order of pairs in a binary relation, and that composition combines two binary relations.

Next we look at functions.

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